Periodic sets and periodic functions.

Arkady Alt.

Short preface.

The basis of this article is the special course that was read by the author for university students in 1989. The main purpose of this special course was the consideration of periodicity conditions of a sums and products of periodic functions.

In the definition of periodic functions important role is played the concept of a periodic set, with we start our narration.

I. **Periodic sets**

Definition 1.

We say that set A is periodic if there is number $\tau \neq 0$ such that $a \in A$ implies $a \pm \tau \in A$ (shortly $A \pm \tau \subset A$)

(empty set $\varnothing$ by definition is periodic and its period is any real nonzero number)

Remark.

If A is periodic set with period τ then:

1. $\mathbb{R} \diagdown A$ is periodic with period τ as well;

2. If $B \subset A$ then $B + \tau\mathbb{Z} \subset A$ and $B + \tau\mathbb{Z}$ is periodic set.

Proof.

1. Let $x \in \mathbb{R} \diagdown A$ then $x \pm \tau \in \mathbb{R} \diagdown A$ because otherwise if $x \pm \tau \notin \mathbb{R} \diagdown A$ then $x \pm \tau \in A$ and, therefore, $x = (x \pm \tau) \mp \tau \in A,$ that contradict to $x \in \mathbb{R} \diagdown A.$

2. Immediatelly follow from definition.

Examples of a periodic sets.

1.$\mathbb{R}$ is obviously periodic set;

2.Let $a, b \in \mathbb{R}$ and $b \neq 0$ then $a + b\mathbb{Z} := \{a + nb \mid n \in \mathbb{Z}\}$ is periodic set with period b;

3. $\mathbb{R} \diagdown (a + b\mathbb{Z})$ is periodic with period b;

4. Let $a, b \in \mathbb{R}$ and $a, b \neq 0$ then $a\mathbb{Z} + b\mathbb{Z} := \{am + bn \mid n, m \in \mathbb{Z}\}$ is periodic set with periods a and b.Furthermore, any element of this set is also its period. Indeed, let $\tau := am_0 + bn_0$.Then

$am + bn \pm (am_0 + n_0 b) = a(m \pm m_0) b + (n \pm n_0)$.

If $\dfrac{a}{b} \in \mathbb{Q}$ that is $\dfrac{a}{b} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ then $\tau = aq = pb$ is period as well.

If $\dfrac{a}{b} \notin \mathbb{Q}$ then $a\mathbb{Z} + b\mathbb{Z}$ is everywhere dence and, therefore, contain any small enough positive τ which is period of this set as well. Hence, in that case, $\mathbb{R} \diagdown (a\mathbb{Z} + b\mathbb{Z})$ is periodic as well with any such τ as a period. (see **Remark** above).

Examples 2,4 represent closed* periodic sets. Periodic sets in example 3.and set $\mathbb{R} \diagdown (a\mathbb{Z} + b\mathbb{Z})$ represent some open* periodic sets.

Let A be periodic set and let $\mathbb{Z}_* := \mathbb{Z} \diagdown \{0\}$. We will denote via $\Omega(A)$ set of all periods of set A.

If $\tau \in \Omega(A)$ then $\tau\mathbb{Z}_* \in \Omega(A)$.(math induction);

If $\tau_1, \tau_2 \in \Omega(A)$ then $\tau_1 + \tau_2 \in \Omega(A)$ and furtermore, $\tau_1 \mathbb{Z}_* + \tau_2 \mathbb{Z}_* \in \Omega(A)$;
$\Omega(A)$ is periodic as well with any own element as a period.
(because for any $\omega \in \Omega(A)$ and any $\tau \in \Omega(A)$ we have $\omega \pm \tau \in \Omega(A)$)
so $\Omega(\Omega(A)) = \Omega(A)$.

Let $\Omega_+(A) := \{\tau \mid \tau \in \Omega(A) \text{ and } \tau > 0\}$.Since $\Omega(A) \neq \varnothing$ for periodic set A then $\Omega_+(A) \neq \varnothing$ as well.

For $\Omega(A)$ can be two options presented by two following propositions:

Proposition1.

Let $\Omega_+(A)$ contain smallest element. Then $\mu := \min \Omega_+(A)$ we call *main period* of the periodic set A
and $\Omega(A) = \mu \mathbb{Z}_*$.

Proof.

Assume contrary that there is $\tau \in \Omega(A)$ such that $\tau \notin \mu\mathbb{Z} \iff \dfrac{\tau}{\mu} \notin \mathbb{Z} \iff \left\{\dfrac{\tau}{\mu}\right\} \neq 0$.

Then $\dfrac{\tau}{\mu} = \left[\dfrac{\tau}{\mu}\right] + \left\{\dfrac{\tau}{\mu}\right\} \iff \tau = n\mu + \rho$ where $n = \left[\dfrac{\tau}{\mu}\right]$ and $\rho = \mu\left\{\dfrac{\tau}{\mu}\right\}$.

Since $\rho = \tau - n\mu \in \Omega_+(A)$ and $\rho < \mu$. That contrdict to $\mu = \min \Omega_+(A)$.

Proposition2.

Let $\Omega_+(A)$ have no smallest element, that is no $\min \Omega_+(A)$ only $\inf \Omega_+(A)$.
Then $\inf \Omega_+(A) = 0$ and $\Omega_+(A)$ dense in $(0, \infty)$.

Proof.

Let $\mu := \inf \Omega_+(A)$. Then $\mu \notin \Omega_+(A)$. We will prove that $\mu = 0$.

Suppose that $\mu > 0$. Then for any $\varepsilon > 0$ there is $\tau \in \Omega_+(A)$ such that $\mu < \tau < \mu + \varepsilon$.

Since $\Omega_+(A)$ have no smallest period then there is $\tau_1 \in \Omega_+(A)$ such that $\tau_1 < \tau$.

Thus we have $\mu < \tau_1 < \tau < \mu + \varepsilon$ and, therefore, $\tau - \tau_1 < \mu + \varepsilon - \mu = \varepsilon$.

Since $\tau - \tau_1$ is positive period of A then we can conclude that for any $\varepsilon > 0$ there is
$\tau \in \Omega_+(A)$ such that $\tau < \varepsilon$.

In particular for $\varepsilon := \mu$ there is positive period $\tau < \mu$ and that contradict $\mu = \inf \Omega_+(A)$.

Let $(\alpha, \beta) \subset (0, \infty)$ then there is $\tau \in \Omega_+(A)$ such that $\tau < \beta - \alpha$.

Since $\dfrac{\beta}{\tau} - \dfrac{\alpha}{\tau} > 1$ then there is $n \in \mathbb{N}$ such that $\dfrac{\alpha}{\tau} < n < \dfrac{\beta}{\tau} \iff \alpha < n\tau < \beta/$
and since $n\tau \in \Omega_+(A)$ it is mean that $\Omega_+(A)$ dense in $(0, \infty)$.

Operations with periodic sets.

1. Let A, B be periodic sets then what we can say about periodicity of $A + B, A \cup B, A \cap B$?

Let A, B be periodic sets with *main periods* μ_A, μ_B:

If μ_A, μ_B are commensurable, that is $\dfrac{\mu_A}{\mu_B} \in \mathbb{Q}$ then sets $A+B, A\cup B, A\cap B$ all periodic.

Proof.

Let $\dfrac{\mu_A}{\mu_B} = \dfrac{p}{q}, p, q \in \mathbb{N}$ then for $\tau := q\mu_A = p\mu_B$ we have:

$A+B$ is periodic with period τ.Indeed,let $a+b \in A+B$ then $a+b+\tau \in A+B$ because $a+\tau = a+q\mu_A \in A$;

$A\cup B$ is periodic with period τ.Indeed,let $x \in A\cup B$ then we have:

$x \in A \implies x+\tau = x+q\mu_A \in A \subset A\cup B$;

$x \in B \implies x+\tau = x+p\mu_B \in B \subset A\cup B$.

$A\cap B$ is periodic with period τ.Indeed, let $x \in A\cap B$ then

$x+\tau = x+q\mu_A \in A$ and $x+\tau = x+p\mu_B \in B$.Hence, $x+\tau \in A\cap B$.

But if μ_A, μ_B are incommensurable, that is $\dfrac{\mu_A}{\mu_B} \notin \mathbb{Q}$, then $A\cup B$ can be unperiodic set.

Example.

Let $A := \mathbb{Z}, B := \alpha\mathbb{Z}$ then both periodical ($\Omega(A) = \mathbb{Z}_*$ and $\Omega(B) = \alpha\mathbb{Z}_*$) but $A\cup B$ isn't periodic.

Proof.

Assume contrary that $A\cup B$ is periodic with positive period τ.Consider three possible logical cases:

1. $\tau \in \Omega(A) = \mathbb{Z}_*$ Since $b+\tau \notin B$ for any $b \in B$ and $b+\tau \in A\cup B$ remains $b+\tau \in A \iff n\alpha+\tau = a$

for some $n, a \in \mathbb{Z}$ and, therefore, $n\alpha = a-\tau \iff \alpha \in \mathbb{Q}$.That is contradiction;

2. $\tau \in \Omega(B) = \alpha\mathbb{Z}_*$.Since $a+\tau \notin A$ for any $a \in A$ and $a+\tau \in A\cup B$ remains $a+\tau \in B \iff a+\tau = m\alpha$

for some $m, a \in \mathbb{Z}$ and, therefore, $a \in \alpha\mathbb{Z}$.That is the contradiction again.

3. $\tau \notin \Omega(A)\cap\Omega(B)$. Since τ isn't integer then $a+\tau \notin A$ for any $a \in A = \mathbb{Z}$ and then remains $a+\tau \in B$

that is $a+\tau = n\alpha$ for some $n \in \mathbb{Z}$.But then for two distinct elements of A,two integers $a_1 \neq a_2$, we have

$a_1+\tau = n_1\alpha, a_2+\tau = n_2\alpha$,where $n_1, n_2 \in \mathbb{Z}$ and then $a_1-a_2 = \alpha(n_1-n_2) \implies \alpha \in \mathbb{Q}$.

Thus, $\mathbb{Z}\cup\alpha\mathbb{Z}$ isn't periodic.

Since $\mathbb{Z}\cup\alpha\mathbb{Z}$ isn't periodic then $\mathbb{R}\diagdown(\mathbb{Z}\cup\alpha\mathbb{Z}) = \mathbb{R}\diagdown\mathbb{Z}\cap\mathbb{R}\diagdown\alpha\mathbb{Z}$ isn't periodic as well.

Indeed, if $\mathbb{R}\diagdown\mathbb{Z}\cap\mathbb{R}\diagdown\alpha\mathbb{Z}$ is perodic, that is periodic $\mathbb{R}\diagdown(\mathbb{Z}\cup\alpha\mathbb{Z})$, then must be periodic

$\mathbb{R}\diagdown(\mathbb{R}\diagdown(\mathbb{Z}\cup\alpha\mathbb{Z})) = \mathbb{Z}\cup\alpha\mathbb{Z}$.

So, it is example of non-periodic intersection of two periodic sets $\mathbb{R}\diagdown\mathbb{Z}$ and $\mathbb{R}\diagdown\alpha\mathbb{Z}$.

And what about $\mathbb{Z}+\alpha\mathbb{Z}$? It is also non-periodic.

Assume contrary that τ is period of $\mathbb{Z}+\alpha\mathbb{Z}$.Then $a_1+b_1\alpha+\tau = c_1+d_1\alpha$ and $a_2+b_2\alpha+\tau = c_2+d_2\alpha$, where $a_i, b_i, c_i, d_i \in \mathbb{Z}, i = 1, 2$.

Hence $a_2-a_1+(b_2-b_1)\alpha = c_2-c_1+(d_2-d_1)\alpha \implies \alpha \in \mathbb{Q}$.

Proposition 3.

Let A, B be two periodic sets such that $B \subset A$ then $\Omega(B \cap A) = \Omega(B)$ and

$\Omega(B \cup A) = \Omega(A)$.

Everywher further we will cosider the functions defined only on open periodic sets.

II. Periodic functions

Definition 2.

Let D be periodic set. We say that function $f : D \longrightarrow \mathbb{R}$ is periodic if there is $\tau \in \Omega(D)$ such that

$f(x+\tau) = f(x)$ for any $x \in D$.

We denote set of all periods of periodic function f via $\Omega(f)$.

Obvious that $\Omega(f) \subset \Omega(D)$

Proposition1.

1. If $\tau_1, \tau_2 \in \Omega_f$ and $\tau_1+\tau_2 \neq 0$ then $\tau_1+\tau_2 \in \Omega(f)$ (shortly $\Omega(f)+\Omega(f) \subset \Omega(f)$);

2. If $\tau \in \Omega_f$ and $n \in \mathbb{Z} \diagdown \{0\}$ then $n\tau \subset \Omega_f$ (shortly $\mathbb{Z} \diagdown \{0\} \cdot \Omega(f) \subset \Omega(f)$).

Proof.

1. Since τ_1, τ_2 are periods of function f then for any $x \in D$ we have

$f(x+(\tau_1+\tau_2)) = f((x+\tau_1)+\tau_2) = f(x+\tau_1) = f(x)$;

2. Let τ is period of f then $f(x) = f((x-\tau)+\tau) = f(x-\tau) = f(x+(-\tau))$. So, $-\tau \in \Omega(f)$.

Also, since $f(x+(n+1)\tau) = f((x+n\tau)+\tau) = f(x+n\tau)$ then by Math Induction

we get $f(x+n\tau) = f(x)$ for any natural n. Similarly, for period $-\tau$ we get $f(x+n(-\tau)) = f(x)$.

So, $f(x+n\tau) = f(x)$ for any integer n.

Let $\Omega_+(f) := \{\tau \mid \tau \in \Omega(f) \text{ and } \tau > 0\}$. Since $\pm\tau \in \Omega(f)$ then $|\tau| \in \Omega_+(f)$ and, therefore, $\Omega_+(f) \neq \varnothing$.

Proposition 2. If $\Omega_+(f)$ has a smallest period, let it be τ_* then $\Omega(f) = \tau_*\mathbb{Z}$,

that is for any $\tau \in \Omega(f)$ there is an integer $n \neq 0$ such that $\tau = n\tau_*$.

Proof.

Assume contrary that there is $\tau \in \Omega(f)$ such that $\dfrac{\tau}{\tau_*} \notin \mathbb{Z} \iff \left\{\dfrac{\tau}{\tau_*}\right\} \neq 0$.

Then $\dfrac{\tau}{\tau_*} = \left[\dfrac{\tau}{\tau_*}\right] + \left\{\dfrac{\tau}{\tau_*}\right\} \iff \tau = n\tau_* + \rho$ where $n = \left[\dfrac{\tau}{\tau_*}\right]$ and $\rho = \tau_*\left\{\dfrac{\tau}{\tau_*}\right\}$.

Since $\rho = \tau - n\tau_* \in \Omega_+(f)$ and $\rho < \tau_*$. That contrdict to $\tau_* = \min\Omega_+(f)$.

In the case $\tau_* = \min\Omega_+(f)$ we call τ_* <u>main period</u> of function f.

Proposition 3.

If $\Omega_+(f)$ have no smallest period then for any $\varepsilon > 0$ there is $\tau \in \Omega_+(f)$ such that $\tau < \varepsilon$

and furtheremore, $\Omega_+(f)$ dense in $(0,\infty)$ ($\Longrightarrow$ $\Omega(f)$ everywhere dense).

Proof.

Let $\tau_* := \inf \Omega_+(f)$. Then $\tau_* \notin \Omega_+(f)$. We will prove that $\tau_* = 0$.

Suppose that $\tau_* > 0$. Then for any $\varepsilon > 0$ there is $\tau \in \Omega_+(f)$ such that $\tau_* < \tau < \tau_* + \varepsilon$.

Since $\Omega_+(f)$ have no smallest period then there is $\tau_1 \in \Omega_+(f)$ such that $\tau_1 < \tau$.

Thus we have $\tau_* < \tau_1 < \tau < \tau_* + \varepsilon$ and, therefore, $\tau - \tau_1 < \tau_* + \varepsilon - \tau_* = \varepsilon$.

Since $\tau - \tau_1$ is positive period of f then for any $\varepsilon > 0$ there is $\tau \in \Omega_+(f)$ such that $\tau < \varepsilon$.

In particular for $\varepsilon := \tau_*$ there is positive period $\tau < \tau_*$ and that contradict $\tau_* = \inf \Omega_+(f)$.

Let $(\alpha, \beta) \subset (0, \infty)$ then there is $\tau \in \Omega_+(f)$ such that $\tau < \beta - \alpha$.

Since $\dfrac{\beta}{\tau} - \dfrac{\alpha}{\tau} > 1$ then there is $n \in \mathbb{N}$ such that $\dfrac{\alpha}{\tau} < n < \dfrac{\beta}{\tau} \iff \alpha < n\tau < \beta/$

and since $n\tau \in \Omega_+(f)$ it is mean that $\Omega_+(f)$ dense in $(0,\infty)$.

Proposition 4.

If $f : D \longrightarrow \mathbb{R}$ is periodic with period τ and $a \neq 0, b \notin \Omega(f)$ then function $x \longmapsto f(ax+b) : a^{-1} \cdot (D - b) \longrightarrow \mathbb{R}$ is periodic as well with period τa^{-1}.

Proof.

$a^{-1} \cdot D$ is periodic set with period τa^{-1}.Indeed, let $x \in a^{-1} \cdot (D - b)$ then $x = a^{-1}(t - b), t \in D$ and $x \pm \tau a^{-1} = a^{-1}((t-b) \pm \tau) = a^{-1}((t \pm \tau) - b) \in a^{-1} \cdot (D - b)$

because $t \pm \tau \in D$;

Also, $f\left(a\left(x + \tau a^{-1}\right) + b\right) = f\left(a\left(a^{-1}(t-b) + \tau a^{-1}\right) + b\right) == f((t-b) + \tau + b) = f(t+\tau) = f(t) = f(ax+b)$.

Proposition 5.

If $f(x)$ is periodic with period τ and differentiable then $f'(x)$ is periodic as well with period τ.

Proof.

We have $f'(x+\tau) = \lim\limits_{h \to 0} \dfrac{f((x+h)+\tau) - f(x+\tau)}{h} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x)$.

Theorem1.

Let function $f : D \longrightarrow \mathbb{R}$ is periodic and continuous at least in one point $a \in D$.

Then f have no smallest positive period iff $f(x)$ is a constant function.

Proof.

Let f have no smallest positive period and $x \in D$ be any. Since f is continuous in a

then for any $\varepsilon > 0$ there is $\delta > 0$ that for any $u \in D$ inequality $|u - a| < \delta$ yield $|f(u) - f(a)| < \varepsilon$.

From the other hand since $\Omega(f)$ everywhere dense (**Proposition 3)** there is $\tau \in \Omega(f)$

such that $|(a - x) - \tau| < \delta \iff |a - (x + \tau)| < \delta$.Then $|f(x + \tau) - f(a)| < \varepsilon \iff |f(x) - f(a)| < \varepsilon$.

Since inequality $|f(x) - f(a)| < \varepsilon$ holds for any $\varepsilon > 0$ then $f(x) = f(a)$.

If $f(x)$ is constant function the it is obviously continuous and any real $\tau \neq 0$ is a period.

Then $\Omega_+(f) = (0, \infty)$ and have no smallest positive period.

Corollary.(Follow immediatelly from the Theorem)

Let function $f : D \longrightarrow \mathbb{R}$ is periodic and continuous at least in one point $a \in D$.

If $f(x)$ isn't a constant function then $f(x)$ has smallest positive period τ_* and then

$\Omega(f) = \tau_* \cdot \mathbb{Z} \diagdown \{0\}$.

Remark. Another,direct proof of the Corollary.

First note that for any $a \in \mathbb{R}$ and any $b \in \mathbb{R} \diagdown \{0\}$ there are unique pair (k, r) such that

$k \in \mathbb{Z}$, $0 \leq \rho < |b|$ and $a = kb + \rho$.

Indeed, .since $\dfrac{a}{b} = \left[\dfrac{a}{b}\right] + \left\{\dfrac{a}{b}\right\}$ then a has unique representation $a = kb + \rho$,where $k := \left[\dfrac{a}{b}\right]$,

$\rho := b\left\{\dfrac{a}{b}\right\} \in [0, b)$.

Assume that $f(x)$ isn't a constant but set $\Omega_+(f)$ have no smallest element

Let $x \in D$ be any and let τ_1 be ay positive period.

Then $x - a = k_1\tau_1 + \rho_1$, where $k_1 = \left[\dfrac{x - a}{\tau_1}\right]$, $\rho_1 := \tau_1\left\{\dfrac{x - a}{\tau_1}\right\} \in [0, \tau_1)$.

If $\rho_1 = 0$ then $x = a + k_1\tau_1 \implies f(x) = f(a + k_1\tau_1) = f(a)$;

If $\rho_1 > 0$ then since $\Omega_+(f)$ have no minimal period there is $\tau_2 \in \Omega_+(f)$ that $\tau_2 < \rho_1$.

Then $\rho_1 = k_2\tau_2 + \rho_2$, where $0 \leq \rho_2 < \tau_2 < \rho_1 < \tau_1$.

If $\rho_2 = 0$ then $x - a - k_1\tau_1 = \rho_1 = k_2\tau_2 \implies x = a + k_1\tau_1 + k_2\tau_2$ and, therefore,

$f(x) = f(a + k_1\tau_1 + k_2\tau_2) = f(a)$.

If $\rho_2 > 0$ we can continue this process.

Assume that we aready have representation $x = a + k_1\tau_1 + k_2\tau_2 + ... + k_n\tau_n + \rho_n$, where

$0 \leq \rho_n < \tau_n < \rho_{n-1} < \tau_{n-1} < ... < \tau_2 < \rho_1 < \tau_1$.

If $\rho_n = 0$ then as before we obtain $f(x) = f(a + k_1\tau_1 + ... + k_n\tau_n) = f(a)$.

Consider now case when process is infinite.

Since $\inf \Omega_+(f) = 0$ we can provide $\lim\limits_{n\to\infty} \rho_n = 0$ if we claim on $n-th$ step that

$\tau_n < \min\{\tau_{n-1}, 1/n\}$ for each $n \in \mathbb{N}$.

Hence, $f(x) = f(a + k_1\tau_1 + ... + k_n\tau_n + \rho_n) = f(a+\rho_n)$ and $f(x) = \lim\limits_{n\to\infty} f(a+\rho_n) = f(a)$

because f is continuous in a.So, we obtaned $f(x) = f(a)$ for any $x \in D$ that is contradiction

with $f(x)$ isn't a constant function.

Exersices which represents examples of periodic functions.

1. $\sin x, \cos x$ are periodic on $\mathbb{R}$. Prove that 2π is main period;

2. $\tan x$ is periodic on $\mathbb{R}\diagdown\{\pi/2 + n\pi \mid n \in \mathbb{Z}\}$; $\cot x$ periodic on $\mathbb{R}\diagdown\{n\pi \mid n \in \mathbb{Z}\}$.

Prove that π is main period for both.

3.Prove that $\{x\}$ is periodic on $\mathbb{R}$ with main period 1;

4. Let $d(x) := \begin{cases} 1 \text{ if } x \in \mathbb{R}\diagdown\mathbb{Q} \\ 0 \text{ if } x \in \mathbb{Q} \end{cases}$ (Dirihlet Function). Prove that $d(x)$ is periodic, everywhere discontinuous function and $\Omega(d) = \mathbb{Q}\diagdown\{0\}$.($\mathbb{Q}$ everywhere dense).

5. Prove that function $\{x-a\}\, d(x)$ is periodic on $\mathbb{R}$ with main period 1 and continuous only in points $a+n, n \in \mathbb{Z}$..

Problems with solutions.

Problem 1.

Find all continuous functions f that satisfy equations $f(x) = f(x+1) = f\left(x+\sqrt{2}\right)$.

Solution.

Since $f(x+1) = f(x)$ and $f\left(x+\sqrt{2}\right) = f(x)$ then $f(x)$ has simulteneously period 1 and period $\sqrt{2}$.

From the other hand $f(x)$ as contfunctinuos function have two options, namely $f(x)$ is constant

function or if it isn't constant and then $f(x)$ has smallest positive period τ_*.

In that case $1 = n\tau_*$ and $\sqrt{2} = m\tau_*$ for some integer n, m.Then $\sqrt{2} = \dfrac{m}{n} \in \mathbb{Q}$,that is the contradiction.

Thus remains that $f(x)$ is a constant function.

Another solution.

Since 1 and $\sqrt{2}$ are periods of $f(x)$ then $m+n\sqrt{2}$ for any $m, n \in \mathbb{Z}\diagdown\{0\}$ is period as well.

Since set $\left\{m+n\sqrt{2} \mid m, n \in \mathbb{N}\right\} \subset \Omega_+(f)$ by* Kronecker Theorem dense in $(0,\infty)$ then $\Omega_+(f)$

is dense in $(0,\infty)$ and, therefore, then by Theorem $f(x)$ is a constant function.

Problem 2.

Prove that $\sup\left(\sin\pi x+\sin\pi\sqrt{2}x\right)=2.$

Solution.

It is obwious that $\sin\pi x+\sin\pi\sqrt{2}x\leq 2.$Also note that $\sin\pi x+\sin\pi\sqrt{2}x=2\iff$

$$\begin{cases}\sin\pi x=1\\ \sin\pi\sqrt{2}x=1\end{cases}\iff\begin{cases}x=1/2+2n,n\in\mathbb{Z}\\ x=\dfrac{1/2+2m}{\sqrt{2}},m\in\mathbb{Z}\end{cases}\implies 1/2+2n=\frac{1/2+2m}{\sqrt{2}}\implies\sqrt{2}\in\mathbb{Q}.$$

Thus, $\sin\pi x+\sin\pi\sqrt{2}x<2$ for any real x.

By replacing x with $1/2+2n$ we obtain $0<1-\sin\pi\sqrt{2}\left(1/2+2n\right)=$
$1-\cos\left(\pi\sqrt{2}\left(1/2+2n\right)-\dfrac{\pi}{2}-2m\pi\right)=2\sin^2\left(\dfrac{\pi\sqrt{2}\left(1/2+2n\right)}{2}-\dfrac{\pi}{4}-m\pi\right).$

Then suffice to prove that there are integers n,m such that
$\left|\dfrac{\sqrt{2}\left(1/2+2n\right)}{2}-\dfrac{1}{4}-m\right|<\varepsilon$ for any $\varepsilon\in(0,\pi/2)$

because then $2\sin^2\left(\dfrac{\pi\sqrt{2}\left(1/2+2n\right)}{2}-\dfrac{\pi}{4}-m\pi\right)<2\sin^2\varepsilon<2\varepsilon^2.$

We have $\left|\dfrac{\sqrt{2}\left(1/2+2n\right)}{2}-\dfrac{1}{4}-m\right|<\varepsilon\iff\left|\sqrt{2}n-m-\dfrac{\sqrt{2}-1}{4}\right|<\varepsilon$

and latter inequality holds for some integers n,m because set $\left\{\sqrt{2}n-m\mid n,m\in\mathbb{Z}\right\}$ dense everywhere by Kronecker Theorem.

Problem 3.

Prove that $\sin x+\sin\sqrt{2}x$ is non-periodic function.

Solution 1.

Suppose that $\sin x+\sin\sqrt{2}x$ is periodic with the period τ.

Then $\sin\left(x+\tau\right)+\sin\sqrt{2}\left(x+\tau\right)=\sin x+\sin\sqrt{2}x\iff$
$\sin\left(x+\tau\right)-\sin x=-\left(\sin\sqrt{2}\left(x+\tau\right)-\sin\sqrt{2}x\right).$

Let $h\left(x\right):=\sin\left(x+\tau\right)-\sin x=-\left(\sin\sqrt{2}\left(x+\tau\right)-\sin\sqrt{2}x\right).$

Then $h\left(x\right)$ is periodic with period τ.But at the same time $h\left(x\right)$ have periods 2π and $\sqrt{2}\pi$.

Note that $\tau\notin 2\pi\mathbb{Z}\diagdown\{0\}$ because otherwise if $\tau\in 2\pi\mathbb{Z}\diagdown\{0\}$ then
$\sin\sqrt{2}\left(x+\tau\right)-\sin\sqrt{2}x=0$ for any x and in paricular if $x=0$.

Then $\sin\sqrt{2}\tau=0\iff\sqrt{2}\tau=k\pi\implies\sqrt{2}\in\mathbb{Z}$ $-$contradiction.

Since continuos function $h\left(x\right)$ isn't constant then it has smallest positive period τ_*.

Then $2\pi=k\tau_*$ and $\sqrt{2}\pi=l\tau_*$ for some integer k,l and, therefore, $\sqrt{2}=\dfrac{k}{l}\in\mathbb{Q}.$

Contadiction!

Solution 2.

Let $h\left(x\right):=\sin x+\sin\sqrt{2}x,$ then $h'\left(x\right)=\cos x+\sqrt{2}\cos\sqrt{2}x$ and $h''\left(x\right)=-\sin x-2\sin\sqrt{2}x$

Assume that $h(x)$ is periodic with τ.Then $h'(x)$ and $h''(x)$ are periodic with period τ.

Since $h(x)+h''(x) = -\sin\sqrt{2}x$ and $\sin\sqrt{2}x$ has main period $\dfrac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ then $\tau = \sqrt{2}m\pi$

because $h'(x) + h''(x)$ has period τ. Similarly, since $2h(x) + h''(x) = \sin x$ we obtain

$\tau = 2n\pi$. Hence, $\sqrt{2}m\pi = 2n\pi \iff \sqrt{2} = \dfrac{m}{n} \in \mathbb{Q}$,that is contradiction.

Remark. By the same way as above can be proved that $\sin x+\sin\alpha x$ is nonperiodic function if α is any irrational number.

Problem 4.

Prove that $\cos x\sin\sqrt{2}x$ non-periodic function.

Solution.

Suppose that $f(x) := \cos x\sin\sqrt{2}x$ is periodic with the period p.

Since $\cos x\sin\sqrt{2}x = \dfrac{\sin\left(\left(\sqrt{2}+1\right)x\right) + \sin\left(\left(\sqrt{2}-1\right)x\right)}{2}$ then $h(x) := 2f\left(\left(\sqrt{2}+1\right)x\right) = \sin x + \sin\left(\left(2\sqrt{2}+3\right)x\right)$ is periodic as well with period $\tau = p\left(\sqrt{2}-1\right)$.

Further we can get the contradiction using idea of **Solution1** or **Solution 2** for **Problem 3.**

Proposition 6.

Let function $f : D \to \mathbb{R}$ is periodic with period $\tau > 0$ and for some $a \in D$ and f is boundeed on some segement $[a, a+\tau] \subset D$ then f boundeed on D.

Proof.

Let $|f(x)| \leq M$ for $x \in [a, a+\tau]$ then for any $x \in D$ there is integer n such that

$x - n\tau \in [a, a+\tau]$.Indeed, $x - a = \tau n + \rho$, where $n := \left[\dfrac{x-a}{\tau}\right]$ and $\rho := \tau\left\{\dfrac{x-a}{\tau}\right\} \in [0, \tau)$.

Then $x - \tau n = a + \rho \in [a, a+\tau)$ and $|f(x)| = |f(x - n\tau)| \leq M$.

Proposition 7.

If function $f(x)$ defined on $\mathbb{R}$ is periodic and continuous on $\mathbb{R}$ then $f(\mathbb{R}) = [m, M]$ where $m := \min\limits_{x\in[a,a+\tau]} f(x)$, $M := \max\limits_{x\in[a,a+\tau]} f(x)$ for any $a \in \mathbb{R}$.

Theorem 2.

Let f_1, f_2 be non constant periodic functions with periods τ_1, τ_2, respectively, such that at least one of these functions continuous and boundeed on its domain and another has at least one point of continuity on $D(f_1) \cap D(f_2)$. Then $f_1 + f_2$ defined on $D(f_1) \cap D(f_2)$ is periodic iff τ_1, τ_2 are commensurable.

Proof.

Assume that f_1 is continuous and boundeed on $D(f_1)$ and f_2 has at least one point of continuity on $D(f_1) \cap D(f_2)$.

Remark 1.

If $D(f_1)$ contain the segment $[a, a+\tau_1]$ fore some $a \in \mathbb{R}$ we don't need claim boundness because by *Proposition* 6 f_1 is boundeed on $D(f_1)$.

Necessity. (by "reductio ad absurdum").

Let τ_1, τ_2 be incommensurable. We may assume that τ_1, τ_2 are main periods for f_1, f_2,respectively (because both function not a constant and both has points of continuity).

Suppose the contrary that $f_1 + f_2$ defined on open set $D := D(f_1) \cap D(f_2)$ is periodic with some period $\tau > 0$. It, in particular, implies that D, being the periodic set with periods τ_1, τ_2,is periodic set with period τ as well. We have

$f_1(x+\tau) + f_2(x+\tau) = f_1(x) + f_2(x) \iff f_1(x+\tau) - f_1(x) = f_2(x) - f_2(x+\tau)$.

Let $h(x) := f_1(x+\tau) - f_1(x) = f_2(x) - f_2(x+\tau)$.

Then τ_1, τ_2 both are periods of $h(x)$.Since $h(x) = f_1(x+\tau) - f_1(x)$ is continuous in D and periodical then $\Omega(h)$ is dense in D because

$$\{n\tau_1 + m\tau_2 \mid n, m \in \mathbb{Z}\} \subset \Omega(h) \text{ is dense in } D.$$

Indeed, the set $\left\{n\tau_1\tau_2^{-1} + m \mid n, m \in \mathbb{Z}\right\}$ dense in $\mathbb{R}$ since $\tau_1\tau_2^{-1} \notin \mathbb{Q}$ and, therefore, for any interval (α, β) there are $n, m \in \mathbb{Z}$ such that

$$\alpha\tau_2^{-1} < n\tau_1\tau_2^{-1} + m < \beta\tau_2^{-1} \iff \alpha < n\tau_1 + m\tau_2 < \beta.$$

Thus, $h(x)$ as continuous function should be a constant function.

Let $h(x) \equiv c, x \in D$, that is $f_1(x+\tau) - f_1(x) = c$ and $f_2(x+\tau) - f_2(x) = -c$ for any $x \in D$.

Then $f_1(x+\tau) - f_1(x) = c, x \in D$ yields $f_1(x+n\tau) - f_1(x) = nc$ for any $n \in \mathbb{N}$ and, therefore,

$$c = \lim_{n\to\infty} \frac{f_1(x+n\tau) - f_1(x)}{n} = 0 \text{ because , } f_1(x) \text{ is boundeed on } D.$$

So, τ is a common period of both functions f_1 and f_2

Then $\tau = m\tau_1$ and $\tau = n\tau_2$ for some natural n, m and, therefore $m\tau_1 = n\tau_2 \implies \dfrac{\tau_2}{\tau_1} \in \mathbb{Q}$

and that is the contradiction.

Sufficiency.

Let τ_1, τ_2 be commensurable that is $\dfrac{\tau_1}{\tau_2} = \dfrac{m}{n}$. Then $\tau := n\tau_1 = m\tau_2$ is common period for both functions and, therefore, is a period of $f_1 + f_2$, defined on the set $D(f_1) \cap D(f_2)$ (which is periodic with period τ).

Remark 2. Sufficiency don't need continuity.

Problem.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and periodical non constant function with main period τ.

Prove that there is the point $x_0 \in \mathbb{R}$ such that $f(x_0 + \tau/2) = f(x_0)$.

Solution.

Let $h(x) := f(x+\tau/2) - f(x)$.Then $h(x+\tau/2) = f(x+\tau) - f(x+\tau/2) = f(x) - f(x+\tau/2) = -h(x)$.

If $h(x) = 0$ for some x then we get such point. If $h(x) \neq 0$ for some x then

$h(x+\tau/2)\cdot h(x) = -h^2(x) < 0$ and, therefore, there is $x_0 \in (x, x+\tau/2)$ such that
$h(x_0) = 0 \iff f(x_0+\tau/2) = f(x_0)$.

Problem.
Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and periodical functions with the same period and $\lim\limits_{x\to\infty}(f(x) - g(x)) = 0$.Prove that $f(x) = g(x)$ for any $x \in \mathbb{R}$.
Solution.
Let τ be any positive period of these functions.
Then $f(x) - g(x) = f(x+n\tau) - g(x+n\tau)$ and, therefore, $f(x) - g(x) = \lim\limits_{n\to\infty}(f(x+n\tau) - g(x+n\tau)) = 0$.

Problem.
Prove that the following sum of periodical functions is non-periodic:
a) $\sin x + \tan\sqrt{2}x; \alpha$
b) $\tan x + \tan\sqrt{2}x$.
a) **Solution 1.**
Since $D(\sin x) = \mathbb{R}, D\left(\tan\sqrt{2}x\right) = \mathbb{R}\diagdown\left\{\dfrac{\pi}{2\sqrt{2}} + \dfrac{n\pi}{\sqrt{2}} \mid n \in \mathbb{Z}\right\}$ and
$D(\sin x) \cap D\left(\tan\sqrt{2}x\right) = \mathbb{R}\diagdown\left\{\dfrac{\pi}{2\sqrt{2}} + \dfrac{n\pi}{\sqrt{2}} \mid n \in \mathbb{Z}\right\}$ then by **Theorem 2.**
$\sin x + \tan\sqrt{2}x$ is unperiodical.
Solution 2.
Let $h(x) := \sin x + \tan\alpha x$ where $\alpha \notin \mathbb{Q}$. Suppose that $h(x)$ periodic with period $\tau > 0$.Since $h(x)$ is differentiable then $h'(x) = \cos x + \alpha\left(1+\tan^2\alpha x\right), h''(x) = -\sin x + 2\alpha^2\tan\alpha x\left(1+\tan^2\alpha x\right)$ are periodic with period τ as well.
Since $h(\tau) = h(0), h'(\tau) = h'(0), h''(\tau) = h''(0)$ then τ satisfy to the system.
Hence

$$\begin{cases} \sin\tau + \tan\alpha\tau = 0 \\ \cos\tau + \alpha\left(1+\tan^2\alpha\tau\right) = 1+\alpha \\ -\sin\tau + 2\alpha^2\tan\alpha\tau\left(1+\tan^2\alpha\tau\right) = 0 \end{cases} \iff \begin{cases} \sin\tau + \tan\alpha\tau = 0 \\ \cos\tau + \alpha\tan^2\alpha\tau = 1 \\ -\sin\tau + 2\alpha^2\tan\alpha\tau\left(1+\tan^2\alpha\tau\right) = 0 \end{cases}.$$

Then $\sin\tau + \tan\alpha\tau + (-\sin\tau) + 2\alpha^2\tan\alpha\tau\left(1+\tan^2\alpha\tau\right) = 0 \iff$
$\tan\alpha\tau\left(1+2\alpha^2\left(1+\tan^2\alpha\tau\right)\right) = 0 \iff \tan\alpha\tau = 0 \iff \alpha\tau = n\pi, n \in \mathbb{Z}$
and, therefore, $\sin\tau = 0 \iff \tau = m\pi, m \in \mathbb{Z}$
Hence, $n\pi = m\pi\alpha \iff \alpha = \dfrac{n}{m} \in \mathbb{Q}$ and that is contradiction.
b) Solution
Let $h(x) := \tan x + \tan\alpha x$ where $\alpha \notin \mathbb{Q}$.
Then $h'(x) = 1 + \tan^2 x + \alpha\left(1+\tan^2\alpha x\right) = 1 + \alpha + \tan^2 x + \alpha\tan^2\alpha x$,
$\tan\tau + \tan\alpha\tau = 0, \tan^2\tau + \alpha\tan^2\alpha\tau = 0$,
$\tan\tau\left(1+\tan^2\tau\right) + 2\alpha^2\tan\alpha\tau\left(1+\tan^2\alpha\tau\right) = 0$.
Since $\tan\alpha\tau = -\tan\tau$ then
$\tan^2\tau + \alpha\tan^2\alpha\tau = \tan^2\tau(1+\alpha) = 0 \iff \tau = n\pi, n \in \mathbb{Z}$ and

$\tan\alpha\tau = 0 \iff \alpha\tau = m\pi, m \in \mathbb{Z}$. Hence, $\alpha = \dfrac{m}{n} \in \mathbb{Q}$ and that is contradiction.

More simple.

Since $\cot x + \cot\alpha x$ isn't periodic (because $D(\cot x + \cot\alpha x) =$
$D(\cot x) \cap D(\cot\alpha x) = \mathbb{R}\diagdown\pi\mathbb{Z} \cap \mathbb{R}\diagdown\dfrac{\pi}{\alpha}\mathbb{Z}$ is non-periodic)
then $\tan x + \tan\alpha x = \cot(\pi/2x - x) + \cot(\pi/2x - \alpha x)$
is non-periodic as well.

There are periodic and non constant functions that have incommensurable periods.

For instance $f(x) = \begin{cases} 1 \text{ if } x \text{ is algebraic number} \\ 0 \text{ if } x \text{ isn't algebraic number} \end{cases}$,
because among algebraic numbers we can find incommensurable numbers.

* 1. **Kronecker's Theorem** **[1]**

a) For any irrational θ set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ dense in $(0,1)$.

b) For any irrational θ set $\{n\theta + m \mid n \in \mathbb{N}, m \in \mathbb{Z}\}$ everywhere dense (dense in $\mathbb{R}$). (that is for any $a \in \mathbb{R}$ and any $\varepsilon > 0$ there are $n \in \mathbb{N}, m \in \mathbb{Z}$ that $|a - (n\theta + m)| < \varepsilon$).

2. Subset $X \subset \mathbb{R}$ is open if for any $x \in X$ there is positive real ε such that $(x - \varepsilon, x + \varepsilon) \in X$

3. Subset $X \subset \mathbb{R}$ is closed if $\mathbb{R}\diagdown X$ is open

1. A.M. Alt. Dense sets and Kronecker's Theorem, Arhimede Mathematical Journal Vol.8,n.1, 2021, p.19-30